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BRIEF COMMUNICATION

THE INERTIAL COUPLING FORCE

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INTRODUCTION

Several years ago prominent scientists had much discussion about the correct form of two-phase flow equations. Mainly two approaches of modelling two-phase flows can be distinguished. One approach, the two-fluid model, considers two-phase flow as a flow of two mutually interacting continua. The mathematical derivation yields a set of equations which can be regarded as a double set of one-phase flow equations with several additional terms describing the interaction (Drew 1983; Van den Akker 1986).

The other approach treats two-phase flow from a suspension point of view. Herein the flow is regarded as one continuum. An equation for the mixture momentum can be derived which is essentially the same as the well-known Navier–Stokes equation for single-phase flow. When the mixture velocity in the suspension equation is separated into the individual velocities of the two phases, the so-called inertial coupling force arises. Soo (1976) and Chao *et al.* (1978) presented a formulation of this inertial coupling force. Furthermore, they suggest a separation of this inertial coupling force into two parts which should be applied in the momentum equations of the individual phases.

In this paper the inertial coupling forces to be applied in the momentum equations of the individual phases are derived. The interphase mass transfer terms are considered too. The expressions found from this strict mathematical derivation differ from those proposed by Soo (1976) and Chao *et al.* (1978). Objections to the equations presented by Soo (1976) and by Chao *et al.* (1978) have been raised by Crowe (1978) and by No (1982). However, these equations did not concern the derivation of the inertial coupling forces.

THE REYNOLDS TRANSPORT THEOREM

In deriving an expression for the inertial coupling forces, use is made of a fundamental theorem of the continuum mechanics, the transport theorem of Reynolds, which reads:

$$\frac{\mathrm{D}\Psi}{\mathrm{D}t} = \int_B \left(\frac{\mathrm{D}\varphi}{\mathrm{D}t} + \varphi \operatorname{div} \vec{v}\right) \mathrm{d}V \qquad [1]$$

This theorem expresses in a moving frame of reference the rate of change, $\mathrm{D}\Psi/\mathrm{D}t$, of some quantity φ integrated over a control volume B which travels with the flow at velocity $\vec{v}$ [see for example Becker *et al.* (1975)]. In the above equation Ψ is defined by:

$$\Psi(t) = \int_B \varphi \, \mathrm{d}V \qquad [2]$$

THE DERIVATION OF THE INERTIAL COUPLING FORCE

When the two-phase flow is regarded as a mixture, the velocity $\bar{v}$ in [1] should be the velocity of the mixture. One should be aware of this consequence of the adoption of the suspension approach for now every quantity φ is transported by the mixture velocity. This mixture velocity, denoted as $\bar{u}_m$, is usually defined as the mass weighted mean of the velocities $(\bar{u}_1, \bar{u}_2)$ of the individual phases [e.g. Chao *et al.* (1978) but also Hinze (1962)]. The relation for $\bar{u}_m$ reads:

$$\overline{u_m} = \frac{\rho_1}{\rho_1 + \rho_2}\overline{u_1} + \frac{\rho_2}{\rho_1 + \rho_2}\overline{u_2} \qquad [3]$$

Herein, ρ denotes a generalized density, which means the material density of the relevant phase multiplied by the volume fraction of that phase. The individual phases are distinguished by the subscripts 1 and 2.

The momentum equation for the first phase can be derived by defining the φ in [1] to be $\rho_1\overline{u_1}$. This means that the transport of the first phase momentum *transported by the suspension* is described by:

$$\frac{D\Psi}{Dt} = \frac{D}{Dt}\int_B \rho_1\overline{u_1}\,dV = \int_B \left(\frac{D\rho_1\overline{u_1}}{Dt} + \rho_1\overline{u_1}\,\mathrm{div}\,\overline{u_m}\right)dV \qquad [4]$$

The variation in time of the first phase momentum within the control volume B should be caused by volume forces and surface forces integrated over the control volume B. Assume that all those forces can be written so that the variation in momentum can be integrated over the same control volume B as all those forces, then [4] may be written as:

$$\int_B \left(\frac{D\rho_1\overline{u_1}}{Dt} + \rho_1\overline{u_1}\,\mathrm{div}\,\overline{u_m}\right)dV = \int_B (.....)\,dV \qquad [5]$$

In this equation the "....." stands for all possible forces influencing the first phase momentum transport. Because B is an arbitrary control volume, [5] may also be written as:

$$\frac{D\rho_1\overline{u_1}}{Dt} + \rho_1\overline{u_1}\,\mathrm{div}\,\overline{u_m} = (.....) \qquad [6]$$

For the derivation of the coupling force, it is not necessary to specify the right-hand side of [6]. Therefore, the left-hand side of [6] will be considered now. Using the definition of the Lagrangian derivative, $D/Dt = \partial/\partial t + \bar{v}\cdot\partial/\partial\bar{x}$, [6] may be written as:

$$\frac{\partial\rho_1\overline{u_1}}{\partial t} + \overline{u_m}\cdot\nabla\rho_1\overline{u_1} + \rho_1\overline{u_1}\nabla\cdot\overline{u_m} = (.....) \qquad [7]$$

Using the relation $(\mathrm{grad}\,\varphi)\cdot\bar{v} + \varphi\,\mathrm{div}\,\bar{v} = \mathrm{div}\,\varphi\bar{v}$, the second and third term of [7] can be joined:

$$\frac{\partial\rho_1\overline{u_1}}{\partial t} + \nabla\cdot\overline{u_m}(\rho_1\overline{u_1}) = (.....) \qquad [8]$$

In index notation the left-hand side of [8] for the velocity in the i-direction reads:

$$\frac{\partial\rho_1 u_{1i}}{\partial t} + \frac{\partial}{\partial x_k}(\rho_1 u_{1i} u_{m,k}) \qquad [9]$$

Here the summation convention should be applied to the index k. Using [3], [9] can be rewritten as:

$$\frac{\partial\rho_1 u_{1i}}{\partial t} + \frac{\partial}{\partial x_k}\left(\rho_1 u_{1i}\frac{\rho_1 u_{1k} + \rho_2 u_{2k}}{\rho_1 + \rho_2}\right) \qquad [10]$$

rewriting:

$$\frac{\partial \rho_1 u_{1i}}{\partial t} + \frac{\partial}{\partial x_k}\left(\frac{\rho_1}{(\rho_1 + \rho_2)} \rho_1 u_{1i} u_{1k} + \rho_2 u_{1i} u_{2k}\right) \quad [11]$$

rewriting:

$$\frac{\partial \rho_1 u_{1i}}{\partial t} + \frac{\partial}{\partial x_k}\left(\frac{\rho_1 + \rho_2 - \rho_2}{(\rho_1 + \rho_2)} \rho_1 u_{1i} u_{1k}\right) + \frac{\partial}{\partial x_k}\left(\frac{\rho_1 \rho_2}{(\rho_1 + \rho_2)} u_{1i} u_{2k}\right) \quad [12]$$

and finally:

$$\frac{\partial \rho_1 u_{1i}}{\partial t} + \frac{\partial}{\partial x_k}(\rho_1 u_{1i} u_{1k}) + \frac{\partial}{\partial x_k}\left(\frac{\rho_1 \rho_2}{(\rho_1 + \rho_2)} u_{1i}(u_{2k} - u_{1k})\right) \quad [13]$$

The first and second terms in [13] are readily identified as the well-known formulation of the advective and convective transport of the first phase momentum. The third term is the inertial coupling term. By interchanging the indices 1 and 2, the relation for the second phase is easily deduced to be:

$$\frac{\partial \rho_2 u_{2i}}{\partial t} + \frac{\partial}{\partial x_k}(\rho_2 u_{2i} u_{2k}) - \frac{\partial}{\partial x_k}\left(\frac{\rho_2 \rho_1}{(\rho_1 + \rho_2)} u_{2i}(u_{2k} - u_{1k})\right) \quad [14]$$

Combining [13] and [14], an expression for the mixture momentum is obtained and for the combined coupling force that should appear in the mixture equation. Using the Lagrangian derivative, the combined expression reads:

$$\frac{\mathrm{D}_1 \rho_1 u_{1i}}{\mathrm{D}t} + \frac{\mathrm{D}_2 \rho_2 u_{2i}}{\mathrm{D}t} - \frac{\partial}{\partial x_k}\left(\frac{\rho_1 \rho_2}{(\rho_1 + \rho_2)} (u_{2i} - u_{1i})(u_{2k} - u_{1k})\right) \quad [15]$$

The first two terms are clearly the contributions of the two phases on the analogy of the classical single-phase momentum equation. Note that the Lagrangian timescale for the transport of $\rho \overline{u_1}$ is not necessarily equal to the timescale for $\rho \overline{u_2}$. In [4] the Lagrangian timescale was related to the transport of the mixture with velocity $\overline{u_m}$. The third term is the combined coupling force. This term is exactly the same as the one presented by Soo (1976) and Chao *et al.* (1978). They obtained the combined coupling force by writing down at once the momentum equation for the suspension. Furthermore, they proposed a separation of this combined term into two terms for the individual momentum equations which lacks a sound basis. Their separated coupling forces differ from the above derived separated coupling forces. In the papers of Soo and Chao no evidence is given that justifies their separation of the combined coupling force. The above derivation, on the contrary, shows clearly how the expression for the separated inertial coupling force is obtained.

THE VALIDITY OF THE MIXTURE APPROACH

Although this paper does not intend to present a thorough discussion on the question which two-phase flow approach should be applied, some remarks may be made. The crucial assumption in the above presented derivation is the use of the mixture velocity in the transport equation [4]. This assumption determines the occurrence of the inertial coupling force. From a mathematical point of view there is no problem regarding the first phase momentum as a quantity of the mixture flow and transport this quantity with the mixture velocity, but from a physical point of view this assumption may be questionable.

A model for dispersed two-phase flow, derived from a physical point of view, has been proposed by Wallis (1989, 1991). According to the model of Wallis the total kinetic energy of the two-phase flow system may exceed the sum of the kinetic energy of the individual phases. This is because the continuous phase has to flow around the dispersed contaminants present in the flow. The total momentum of the system, however, is merely the sum of the momentum of the two phases because in Wallis' approach no mechanisms are present which allow another result. In the suspension model, on the other hand, the inertial coupling force in [15] represents extra momentum above the sum of the momentum of the two phases. This inertial coupling force, however, arises as a straightforward consequence of the suspension approach itself. The model resulting from the approach taken by Wallis (1989) obviously conflicts with the suspension approach.

From a physical point of view, the starting-points of the model of Wallis are more realistic than the above taken approach and therefore the occurrence of inertial coupling forces in the momentum balance equation for two-phase flow might be rejected. Consequently, the suspension approach might be regarded as inferior to the two-fluid approach to model two-phase flows because the definition of a mixture velocity leads to inappropriate results.

THE INTERPHASE MASS TRANSPORT

The interphase mass transfer has also been considered by Soo (1976) and Chao *et al.* (1978). Again their proposed term to model interphase mass transfer in the momentum equations of the individual phases seems to be incorrect. The reason might be that one should realize that these mass transfer terms in the individual momentum equations have nothing to do with inertial coupling as the inertial coupling arises from the non-linear convective momentum transport term as shown in the above presented derivation.

The extra term due to mass transfer can be deduced by manipulating the momentum equation for one phase in the same way as can be done for the single-phase flow momentum equation. A term identical to the left-hand side of the mass balance equation can be extracted. The mass balance for phase one including mass transfer reads:

$$\frac{\partial \rho_1}{\partial t} + \frac{\partial \rho_1 u_{1k}}{\partial x_k} = \Gamma \qquad [16]$$

Here Γ denotes the mass transfer rate between the phases. Rewriting [13], the mass balance term appears in square brackets:

$$\rho_1 \frac{\partial u_{1i}}{\partial t} + \rho_1 u_{1k} \frac{\partial u_{1i}}{\partial x_k} + u_{1i} \left[\frac{\partial \rho_1}{\partial t} + \frac{\partial \rho_1 u_{1k}}{\partial x_k} \right] + \frac{\partial}{\partial x_k} \frac{\rho_1 \rho_2}{(\rho_1 + \rho_2)} u_{1i}(u_{2k} - u_{1k}) \qquad [17]$$

It is obvious that the term in brackets equals the rate of mass transfer per unit volume Γ when interphase mass transfer is considered. This results in a term in the first phase momentum equation due to mass transfer which reads:

$$u_{1i}\Gamma \qquad [18]$$

CONCLUSIONS

This paper presents mathematical derivations of the interphase coupling terms for mass and momentum, which arise in the two-phase flow equations when the suspension approach is followed. Although the interphase coupling terms presented by Soo (1976) and Chao *et al.* (1978) for the suspension momentum equations are correct, this paper shows that their inertial coupling forces and interphase mass transfer terms which appear in the momentum equations of the individual phases are not correct. It has been discussed that the occurrence of the inertial coupling forces seems to be incorrect from a physical point of view, and consequently the suspension approach might be concluded to be inferior to the two fluid approach.

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